INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS
Int. J. Numer. Anal. Meth. Geomech., 2001; **25**:841–851 (DOI: 10.1002/nag.151)

Short Communication

Calculation of bearing capacity of a strip footing using an upper bound method

Yu-Jie Wang[1], Jian-Hua Yin[1,*,†] and Zu-Yu Chen[2]

[1] *Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China*
[2] *China Institute of Water Resources and Hydropower Research, P.O. Box 366, Beijing 100044, (also Faculty of Civil Engineering, Tsinghua University) People's Republic of China*

1. INTRODUCTION

The upper bound theorem of plasticity as a useful technique to solve the geotechnical problems has been applied widely and extensively to bearing capacity analysis [1–7]. Using the upper- bound theorem, Chen [1] determined the three factors of bearing capacity individually by assuming a reasonable failure mechanism. Michalowski [6] and Soubra [7] presented alternative upper bound methods to determine the three bearing capacity factors by using optimization techniques.

In practice, the ultimate bearing capacity is obtained by the superposition of influences of three contributions of cohesion c, surcharge load q and unit weight of soil γ. Chen [1], Michalowski [6], Soubra [7] and Griffiths [8], have found that the ultimate bearing capacity obtained by considering the joined influences of the three factors for cohesion, surcharge load and unit weight of soil is greater than that obtained by superposition of the influences using the three factors determined individually. Michalowski [6] presents that the bearing capacity factors depend not only the internal friction angle ϕ, but also other material parameters, when the joined influences of cohesion c, surcharge load q and unit weight of soil γ are considered simultaneously.

Donald and Chen [9] proposed an upper-bound method to study the stability of the slopes. This method is extended in this paper to study the bearing capacity of a footing with the associated computer program modified. The objective of this paper is to seek the most reasonable failure mechanism by using the upper-bound approach and an optimization method for the calculation of the bearing capacity of a strip footing on soils without or with the joined influences of the cohesion, surcharge load and soil weight. The three bearing capacity factors are obtained

* Correspondence to: Jian-Hua Yin, Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China.
† E-mail: cejhyin@polyu.edu.hk

Contract grant sponsor: University Grants Committee of the Hong Kong SAR Government of China, and the Hong Kong Polytechnique University, Contract grant number: RGC grant (PolyU 5065/97E).
Contract grant sponsor: China National Natural Science Foundation; contract grant number: 590679013

Received 2 March 2000
Revised 30 November 2000

by considering both joined influence and individual influence in two different approaches. The values of the three factors obtained in the two approaches are compared to each other and to the published results. The critical slip lines (or failure mechanisms) for the joined influences and individual influences are presented and discussed.

2. GENERAL EQUATION OF THE BEARING CAPACITY PROBLEM

Unless specially mentioned in this paper, a strip footing is located on a homogenous and rigid-plastic soil medium under a plane strain condition. The soil above the footing base is replaced with a surcharge load. The ultimate bearing pressure q_u of the footing is equal to the ultimate load that the foundation soil can hold at the state of incipient failure divided by the strip footing area and can be generally expressed in the form (Terzaghi [10])

$$q_u = cN_c + qN_q + 0.5\gamma BN_\gamma \tag{1}$$

where N_c, N_q, N_γ are three bearing capacity factors related to the cohesion c, the surcharge pressure q and the unit weight of the soil γ, respectively.

3. THE UPPER-BOUND THEOREM FOR BEARING CAPACITY ANALYSIS

3.1. Multi-wedge discretization system

Donald and Chen [9] proposed a multi-wedge discretization system for slope stability analysis. The same discretization mode is extended in this paper for bearing capacity analysis. The plastic zone above the assumed slip surface is divided into a number of wedges with inclined interfaces. Each of wedges moves as a rigid body. The plastic energy dissipation occurs at the interfaces between adjoining wedges and at the base of wedges. Figure 1 shows a n-wedge discretization system, in which only one wedge ABC is beneath the strip footing.

It is assumed that the magnitude of velocity of the first left wedge (No. 1 in Figure 1) in the overall failure mechanism is unit since the absolute value of the velocity has no influence on the final results. The velocity direction of wedge No. 1 is inclined at an angle of ϕ_1 to the base of the wedge. If the magnitude and direction of the first wedge is known, the absolute velocities of other wedges and relative velocities at interfaces can be easily determined from velocity analysis [9]. Thus, the overall kinematically admissible velocity field within assumed failure mechanism could be determined.

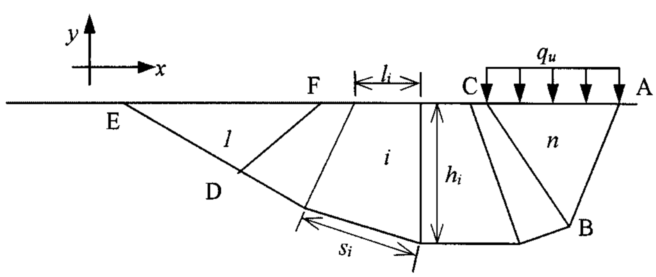

Figure 1. Failure mechanism for the bearing capacity analysis.

3.2. *Ultimate bearing capacity* q_u *and general bearing capacity factors* N_c, N_q and N_γ

According to the *n*-wedge discretization of soil mass under the strip footing and the kinematically admissible velocity field determined previously, the work-energy balance equation for upper bound analysis is written as

$$\sum_{i=1}^{n} \dot{D}_i^s + \sum_{i=1}^{n-1} \dot{D}_i^j = \sum_{i=1}^{n-1} \dot{W}_q + \sum_{i=1}^{n} \dot{W}_\gamma + q_u^* B V^* \quad (2)$$

where $\sum_{i=1}^{n} \dot{D}_i^s$ is the summation of rate of energy dissipation along the base of each wedge and $\sum_{i=1}^{n-1} \dot{D}_i^j$ is the summation of rate of energy dissipation along the interface between wedges. The terms $\sum_{i=1}^{n-1} \dot{W}_q$, $\sum_{i=1}^{n} \dot{W}_\gamma$ and $q_u^* B V^*$ are the rates of work done by the surcharge load, the weight of each wedge and the pressure acting on the footing, respectively. By calculating each term in Equation (2) according to Donald and Chen [9], the general equation of bearing capacity of footing using the upper-bound method can be given as

$$q_u^* = \left\{ \sum_{i=1}^{n} (c_i s_i V_i \cos\phi_i) + \sum_{i=1}^{n-1} (c_i^j h_i V_i^j \cos\phi_i^j) - q \sum_{i=1}^{n-1} [l_i \sin(\alpha_i - \phi_i) V_i] - \sum_{i=1}^{n} [W_i \sin(\alpha_i - \phi_i) V_i] \right\} \Big/ [B V_n \sin(\alpha_n - \phi_n)] \quad (3)$$

where the superscript '*' in q_u^* denotes that the pressure on the footing is determined based on an assumed (or trial) failure mechanism. The real bearing capacity of footing, that is, the ultimate bearing capacity is obtained using an optimization method to search for the minimum value of q_u^* in a large number of reasonable failure mechanisms. In Equation (3), c_i and ϕ_i are the cohesion and friction angle of soil on the base of Wedge No. *i* respectively. The c_i^j and ϕ_i^j are the cohesion and friction angle on the interface between wedges No. *i* and No. $i-1$. The l_i, s_i, h_i are the length of the topside, the base and interface of wedge No. *i*, respectively.

If the water pressure is not considered, the comparison of Equation (3) with Equation (1) indicates that the three bearing capacity factors given by

$$N_c^* = \left[\sum_{i=1}^{n} (s_i V_i \cos\phi_i) + \sum_{i=1}^{n-1} (h_i V_i^j \cos\phi_i^j) \right] \Big/ [B V_n \sin(\alpha_n - \phi_n)] \quad (4)$$

$$N_q^* = \left[-\sum_{i=1}^{n-1} l_i \sin(\alpha_i - \phi_i) V_i \right] \Big/ [B V_n \sin(\alpha_n - \phi_n)] \quad (5)$$

$$N_\gamma^* = \left[-\sum_{i=1}^{n} (W_i \sin(\alpha_i - \phi_i) V_i) \right] \Big/ [V_n \sin(\alpha_n - \phi_n) 0.5\gamma] \quad (6)$$

The final bearing capacity factors N_c, N_q and N_γ are the values of N_c^*, N_q^*, N_γ^* which make q_u^* minimum [11, 12]. It should be noted that the bearing capacity calculated by Equation (3) is equivalent to the bearing capacity for a rough footing on soil, that is, there is no sliding between the footing and the soil [7].

4. NUMERICAL SOLUTIONS OF N_c, N_q, N_γ AND q_u

Previous researchers normally find three factors N_c, N_q and N_γ one by one with a final failure mechanism for each influence of cohesion *c*, surcharge *q* and unit weight of soil γ. As matter of fact, these influences are coupled [6, 8]. Furthermore, the final failure mechanism of the coupled influences may not be the same as that for each individual influence.

Without the influence of water pressure, Equation (3) based on the upper-bound theorem can be used readily to seek the most reasonable failure mechanism for the joined (or coupled) influence or individual influence. To consider the joined influences, the cohesion c, surcharge q, and unit weight of soil γ are non-zero simultaneously. The final failure mechanism is determined by minimizing Equation (3). Once the final failure mechanism is determined, the bearing capacity factors are also determined using Equations (4)–(6), which correspond to the minimal value of the bearing capacity of a footing q_u. The above method is called Method 1, the corresponding bearing capacity factors are denoted as $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ (called joined bearing capacity factors).

To obtain the conventional bearing capacity factors N_c, N_q, N_γ individually, for example, N_c, q and γ are set to be zero in Equation (3). The factor N_c is then obtained by minimizing Equation (3) with a corresponding final failure mechanism. Similarly, the minimal value of N_q can be obtained by neglecting c and γ. The minimal value of N_γ is obtained by neglecting c and q. The bearing capacity factors determined individually in the above way are denoted as N_c, N_q, N_γ, which are refereed to as individual bearing capacity factors. The overall bearing capacity of footing q_u is determined by the superposition method. Herein, the method of determining the individual bearing capacity factors is called Method 2.

The relationship of the joined bearing capacity factors $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ and the individual factors N_c, N_q, N_γ and the difference in the associated final failure mechanisms are studied in this paper. A strip rough footing with width of 1 m rests on a soil with 10 kN/m^2 of cohesion, 30° of friction angle, 18.0 kN/m^3 of unit weight, and 10 kN/m^2 of surcharge pressure [7]. The minimal values of individual bearing capacity factors N_c, N_q, N_γ are found to be 30.2, 18.5, 24.21, respectively with the corresponding failure mechanisms shown in Figure 2(a), (b) and (c).

Using the method of superposition and the value of individual bearing capacity factors N_c, N_q, N_γ, the overall bearing capacity of the footing q_{super} is 704.89 kN/m^2. If considering the joined influences of cohesion, surcharge and unit of weight of soil, the joined bearing capacity factors $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ are found to be 31.48, 18.64, 26.11, respectively. The corresponding failure mechanism is shown in Figure 2(d) with an overall bearing capacity q_u of 732.36 kN/m^2. A comparison of slip surfaces for individual influence of N_c, N_q, N_γ and the joined influence is shown in Figure 2(e). The differences are apparent. All calculated values of the three factors in comparison with those obtained by Soubra [7] are presented in Table I.

It can be seen in Table I that bearing capacity factors $\tilde{N}_c$ and N_c or $\tilde{N}_q$ and N_q are very close. But the value of $\tilde{N}_\gamma$ obtained using Method 1 (considering the joined influence) is 8.61 per cent greater that the value of N_γ obtained using Method 2. The ultimate bearing capacity determined by Method 1 (considering the joined influences) is greater than that by the simple superposition method. Michalowski [6], Soubra [7] and Griffiths [8] had the same observation, which implied that the superposition effect contributed by three individual factors is on the safe side.

To further explore the relation of the joined bearing capacity factors $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ with the individual factors N_c, N_q, N_γ, a strip footing of 1 m width on a homogenous soil is considered. The cohesion c, unit weight of soil γ and the surcharge q are 5, 20 and 10 kN/m^2, respectively. The friction angle ϕ varies from 10 to 45°. The final results are summarized in Table II. The variations of $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ and N_c, N_q, N_γ with friction angle ϕ are shown in Figures 3(a)–(c).

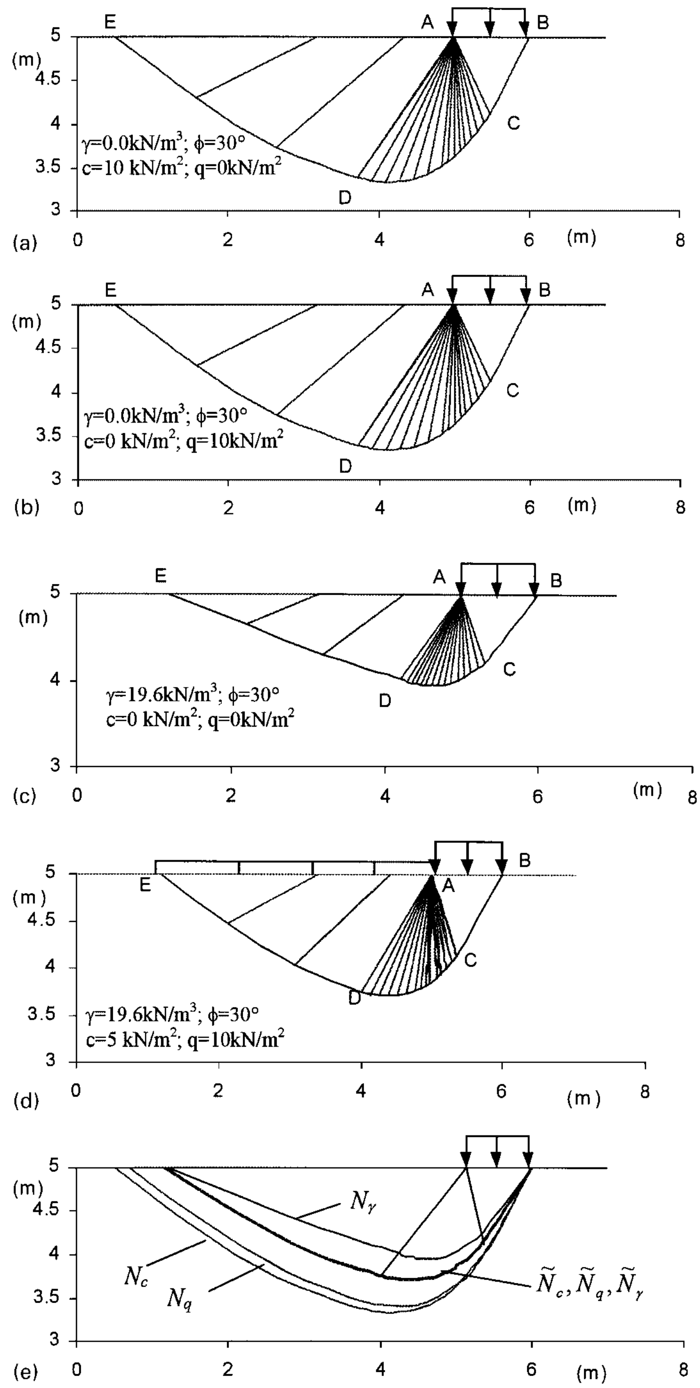

Figure 2. Comparison of failure mechanisms for bearing capacity problem.

Table I. Comparison of bearing capacity factors.

	N_c or $\tilde{N}_c$	N_q or $\tilde{N}_q$	N_γ or $\tilde{N}_\gamma$	q_u or q_{super} (kN/m^2)
Present method 1	31.48	18.64	26.11	736.19
Present method 2	30.20	18.50	24.21	704.89
Soubra's method 1				726.13
Soubra's method 2	30.25	18.46	21.88	680.58

Note. Soubra's method 1 determines the overall bearing capacity of the same footing considering the joined influence. Soubra's method 2 determines the bearing capacity of footing by superposition using individual bearing capacity factors (Soubra [7]). Soubra [7] did not give values of the joined bearing capacity factors.

Table II. Comparison of joined factors $\tilde{N}_c$, $\tilde{N}_q$, $\tilde{N}_\gamma$ with individual factors N_c, N_q, N_γ.

ϕ (deg.)	Present method 1—Joined factors				Present method 2—Individual factors				$(q_u - q_{super})/q_u$ (%)
	$\tilde{N}_c$	$\tilde{N}_q$	$\tilde{N}_\gamma$	q_u (kN/m^2)	N_c	N_q	N_γ	q_{super} (kN/m^2)	
10	9.74	1.98	2.41	96.2	8.64	2.63	1.67	86.2	7.4
11	10.59	2.39	2.61	102.95	9.35	2.74	1.95	93.65	9.9
12	10.98	2.62	3.00	111.16	9.78	3.09	2.26	102.4	8.6
13	11.17	2.98	3.58	121.45	10.24	3.35	2.60	110.7	8.9
14	11.90	3.12	3.93	129.95	10.85	3.63	2.99	120.45	7.9
15	12.47	3.39	4.53	141.54	11.38	4.11	3.53	133.3	6.2
16	13.11	3.94	5.10	155.6	12.05	4.47	3.93	144.25	7.9
17	13.72	4.53	5.71	170.95	12.69	4.83	4.45	156.25	9.4
18	14.47	5.19	6.36	187.35	13.42	5.34	5.19	172.4	8.7
19	15.44	5.61	7.11	204.33	14.21	5.74	5.76	186.05	9.8
20	16.13	5.81	8.01	218.90	15.00	6.47	6.56	205.3	6.6
21	16.91	6.49	9.11	240.57	16.05	7.08	7.45	225.55	6.7
22	18.16	7.21	10.04	263.37	16.86	7.78	8.34	245.5	7.3
23	19.24	8.76	11.27	296.46	18.23	8.59	9.68	273.85	8.3
24	20.33	9.14	12.85	321.6	19.50	9.67	11.16	305.8	5.2
25	21.83	10.74	14.12	357.79	20.80	10.72	12.26	333.8	7.2
26	23.42	11.96	15.80	394.68	22.34	11.80	14.10	370.7	6.5
27	25.24	13.49	17.89	436.87	24.00	13.04	16.07	411.10	6.3
28	26.74	13.6	20.95	479.2	25.80	14.75	18.44	460.9	4.0
29	29.10	16.33	22.97	538.52	27.80	16.42	21.06	513.8	4.8
30	31.70	18.75	25.97	605.72	30.20	18.5	24.21	578.1	4.6
31	34.36	20.61	29.92	677.13	32.86	20.00	28.05	644.8	5.0
32	37.79	24.22	34.38	774.97	35.60	23.25	32.82	738.7	4.9
33	41.29	27.36	39.40	874.02	38.72	26.10	37.51	829.7	5.3
34	45.39	30.85	45.94	994.7	42.50	29.86	44.04	951.5	4.5
35	50.18	35.66	53.60	1143.52	46.50	33.12	50.94	1073.1	6.6
36	54.48	38.73	67.21	1331.32	51.25	37.84	60.60	1240.65	7.4
37	60.40	43.74	78.78	1527.32	55.90	43.39	70.83	1421.70	7.4
38	67.66	52.31	93.76	1797.80	62.57	47.72	84.64	1636.45	9.9
39	75.88	60.76	110.34	2090.41	68.24	54.95	101.78	1908.5	9.5
40	85.56	70.62	129.66	2430.72	75.90	64.90	122.95	2258.00	7.6
41	96.22	83.79	156.93	2888.39	85.61	74.90	148.28	2659.85	8.6
42	106.91	93.45	189.84	3367.45	98.42	87.90	181.39	3185.00	5.7
43	120.59	109.83	231.68	4017.76	109.21	104.47	220.51	3795.85	5.8
44	137.59	131.28	284.24	4843.25	123.69	119.19	270.01	4510.45	7.4
45	157.17	154.69	353.27	5856.52	141.38	142.34	331.22	5442.50	7.8

Note. The strength parameters used in this Table are cohesion $c = 5$ kN/m^2, surcharge load $q = 10$ kN/m^2, and unit weight of soil $\gamma = 20$ kN/m^3. The width of footing $B = 1$ m.

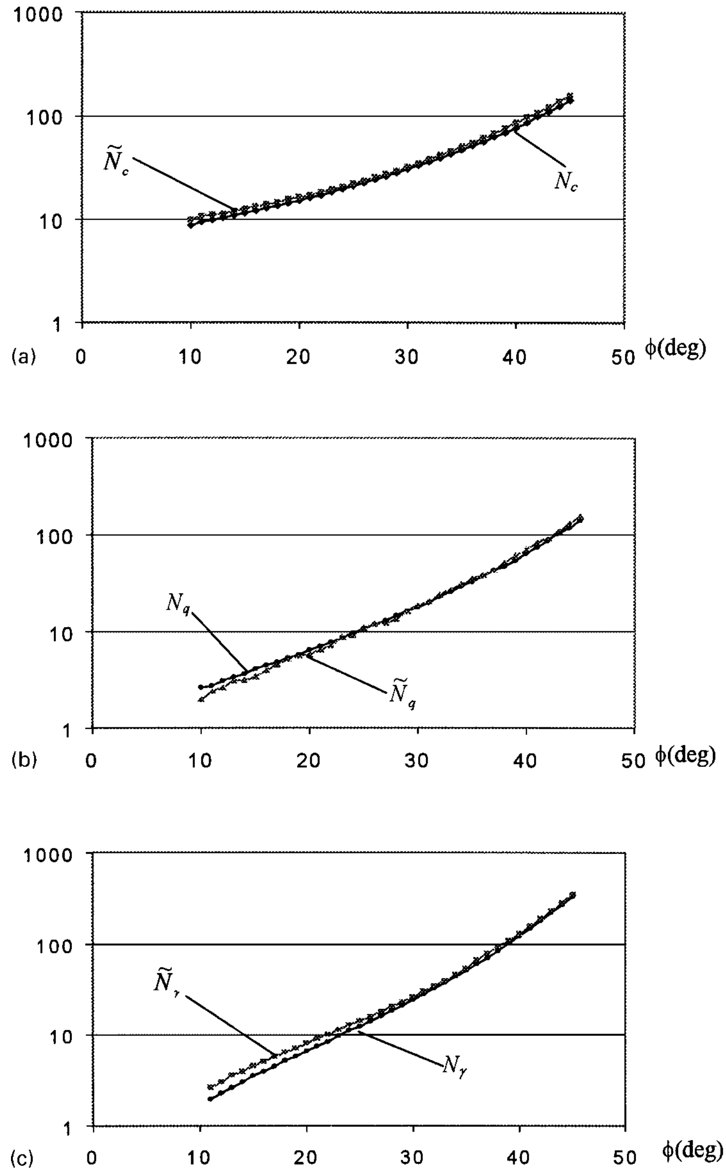

Figure 3. Comparison of bearing capacity factors.

Figure 3 shows that the joined bearing capacity factors $\tilde{N}_c$ and $\tilde{N}_\gamma$ are greater than the individual factors N_c and N_γ. The N_q is almost the same as the individual factor $\tilde{N}_q$. The overall bearing capacity q_u of the footing with joined influences are greater than q_{super} determined by the superposition method. The relative difference is within 10 per cent.

5. COMPARISON OF PRESENT RESULTS WITH OTHER PUBLISHED RESULTS

5.1. N_γ factor

As well known, there have been a great number of solutions for N_γ in the literatures using different methods. The differences among these solutions are substantial. The equations for the N_γ factor given by Terzaghi [10], Meyerhof [13] and Vesic [14] are as follows:

$$N_\gamma(\text{Terzaghi}) = \frac{\tan\phi}{2}\left(\frac{K_{p\gamma}}{\cos^2\phi} - 1\right) \tag{7}$$

for rough footing [10]

$$N_\gamma(\text{Meyerhof}) = (N_q - 1)\tan(1.4\phi) \tag{8}$$

$$N_\gamma(\text{Vesic}) = 2(N_q + 1)\tan\phi \tag{9}$$

In addition, Bolton and Lau [6] proposed the following equation for rough footing by the slip-line method:

$$N_\gamma(\text{Bolton \& Lau}) \approx (N_q - 1)\tan(1.5\,\phi) \tag{10}$$

It should be noted that all the above methods consider the influence of the unit weight of soil individually.

Values of N_γ obtained by Terzaghi [10], Meyerhof [13], Vesic [14] and Bolton and Lau [15] are tabulated in Table III in comparison with the results obtained by present methods. Figure 4 shows a comparison of the N_γ vs. friction angle ϕ. It can be seen that N_γ factor of present Method 2 (not considering the joined influence) is close to that given by Bolton and Lau [15]. However, values of N_γ factor of present Method 1 (considering the joined influence) are greater than all other values as shown in Table III.

Chen [1] gave rigorous upper-bound solution in the framework of the limit analysis theory, based on the Prandtl failure mechanism, which is composed of triangular active wedge beneath the footing, two radial log-spiral shear zones and two triangular passive wedges. Michalowski [6] proposed an upper-bound method considering all joined influences of cohesion, surcharge and soil weight, however, the final factor N_γ is determined by considering the individual influence of unit weight of soil in accordance with other existing proposals. Soubra [7] gave the numerical solution based on the upper bound theorem. The upper-bound solutions given by the present

Table III. Comparison of present N_γ values with those in the literatures.

ϕ (°)	Present method 1	Present method 2	Terzaghi [10]	Meyerhof [13]	Vesic [14]	Bolton and Lau [15]
20	8.01	6.56	5.0	2.87	5.39	5.97
25	14.12	12.26	9.7	6.77	10.88	11.6
30	25.97	24.21	19.7	15.67	22.4	23.6
35	53.60	50.94	42.4	37.15	48.03	51.0
40	129.66	122.95	100.4	93.69	109.41	121.0
45	353.27	331.22	297.5	262.74	271.76	324.0

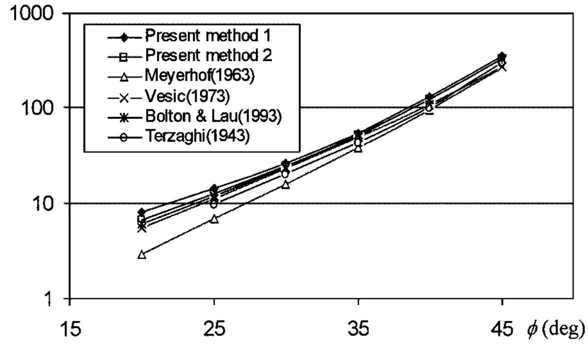

Figure 4. Comparison of present N_γ factor with results of other authors.

Table IV. Comparison of present N_γ with results by other upper-bound methods.

ϕ (°)	Present method 1	Present method 2	Chen [1]	Michalowski [6] (rough)	Soubra's [7] method 2
20	8.01	6.56	5.87	4.47	4.67
25	14.12	12.26	12.4	9.77	10.06
30	25.97	24.21	26.7	21.39	21.88
35	53.60	50.94	60.2	48.68	49.62
40	129.66	122.95	147.0	118.83	120.96
45	353.27	331.22	401.0	322.84	328.88

methods and those given by Chen [1], Michalowski [6] and Soubra [7] are listed in Table IV. Figure 5 shows a comparison of the N_γ vs. friction angle ϕ. The solutions of present Method 2 (not considering the joined influence) are close to those given by Michalowski [6] and Soubra [7]. However solutions of Method 1 (considering the joined influences) are close to those given by Chen [1].

5.2. N_c and N_q factors

In the literature N_c and N_q obtained with different methods are generally expressed in the following forms:

$$N_c = (N_q - 1)\cot\phi \tag{11}$$

$$N_q = e^{\pi \tan\phi}\tan^2(45 + \phi/2) \tag{12}$$

A comparison of present solutions to N_c with results of Equation (11) and Soubra's upper bound numerical solutions [7] is listed in Table V. Similarly, a comparison of N_q is listed in Table VI. It can be seen from Tables V and VI that the N_c and N_q of present Method 2 (not considering the joined influence) are basically identical with those of other methods. However, the N_c and N_q of present Method 1 (considering the joined influence) are slightly greater than those of other methods.

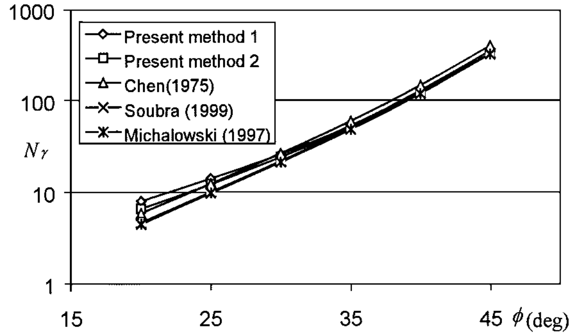

Figure 5. Comparison of present N_γ factor with results of upper bound methods.

Table V. Comparison of present N_c with those of other methods.

ϕ (°)	Present method 1	Present method 2	Equation (11)	Soubra's [7] method 2
20	16.13	15.00	14.83	14.87
25	21.83	20.80	20.71	20.78
30	31.70	30.20	30.13	30.25
35	50.18	46.50	46.33	46.35
40	85.56	75.90	75.25	75.80
45	157.17	141.38	133.73	135.09

Table VI. Comparison of present N_q with those of other methods.

ϕ (°)	Present method 1	Present method 2	Equation (12)	Soubra's [7] method 2
20	5.81	6.47	6.4	6.41
25	10.74	10.72	10.7	10.69
30	18.75	18.16	18.4	18.46
35	35.66	33.12	33.4	33.43
40	70.62	64.90	64.1	64.55
45	154.69	141.38	134.70	135.91

6. CONCLUSIONS

The present methods, which are based on the upper-bound theorem, derive a general bearing capacity equation of strip footing by considering both joined influence and individual influence of cohesion, surcharge load and unit weight of soil. The most reasonable failure mechanism should be searched for among all the kinematically admissible ones by the optimization method. The bearing capacity factors considering both joined influence and individual influence have been obtained and compared to results published in literatures.

Based on the above analysis and results presented, the following conclusions may be drawn:

(a) The failure mechanism of the rough footing bearing capacity problem for the joined influence is different from N_c, N_q, N_γ mechanism considering each individual influence.
(b) The bearing capacity factor considering the unit weight of soil for the joined influence is larger than that for the individual influence. The overall bearing capacity of the footing for the joined influence is larger than that obtained by the superposition method using individual factors.
(c) The results obtained by the present methods starting with a more general failure mechanism are close to those by Michalowski [6] and Soubra [7]. The present methods can be extended to solve more complicated bearing capacity problems.

ACKNOWLEDGEMENTS

The research and preparation of this paper have received financial supports from a RGC grant (PolyU 5065/97E) of the University Grants Committee of the Hong Kong SAR Government of China, the Hong Kong Polytechnic University and the China National Natural Science Foundation grant (No. 590679013). These financial supports are gratefully acknowledged.

REFERENCES

1. Chen WF. *Limit Analysis and Soil Plasticity*. Elsevier: Amsterdam, 1975.
2. Drescher A, Detournay E. Limit load in translational failure mechanisms for associative and non-associative materials. *Geotechnique* 1993; **43**(3):443–456.
3. Michalowski RL. Limit analysis of weak layers under embankments. *Soils and Foundations* 1993; **33**(1):155–168.
4. Michalowski RL. Slope stability analysis: a kinematical approach. *Geotechnique* 1995; **45**(2):283–293.
5. Michalowski RL, Shi L. Bearing capacity of footings over two-layer foundation soils. *ASCE Journal of Geotechnical Engineering* 1995; **121**(5):421–428.
6. Michalowski RL. An estimate of the influence of soil weight on bearing capacity using limit analysis. *Soils and Foundations* 1997; **37**(4):57–64.
7. Soubra AH. Upper-bound solutions for bearing capacity of foundations. *ASCE Journal of Geotechnical and Geoenvironmental Engineering* 1999; **125**(1):59–68.
8. Griffiths DV. Computation of bearing capacity factors using finite elements. *Geotechnique* 1982; **32**(3):195–202.
9. Donald IB, Chen ZY. Slope stability analysis by the upper bound approach: fundamentals and methods. *Canadian Geotechnical Journal* 1997; **34**(6):853–862.
10. Terzaghi K. *Theoretical Soil Mechanics*. Wiley: New York, 1943.
11. Chen ZY. Random trials used in determining global minimum factors of safety of slopes. *Canadian Geotechnical Journal* 1992; **29**(2):225–233.
12. Chen ZY, Shao CM. Evaluation of minimum factor of safety in slope stability analysis. *Canadian Geotechnical Journal* 1988; **25**(4):735–748.
13. Meyerhof GG. The ultimate bearing capacity of foundations. *Geotechnique* 1951; **2**:301–331.
14. Vesic AS. Analysis of ultimate loads of shallow foundations. *JSMFD ASCE* 1973; **99**(1):45–73.
15. Bolton MD, Lau CK. Vertical bearing capacity factors for circular and strip footings on Mohr-Coulomb soil. *Canadian Geotechnical Journal* 1993; **30**(4):1024–1033.
16. Frydman S, Burd HJ. Numerical studies of bearing-capacity factor N_γ. *ASCE Journal of Geotechnical and Geoenvironmental Engineering* 1997; **123**(1):20–29.
17. Davis EH. Theories of plasticity and the failure of soil masses. In: *Soil Mechanics* Lee IK. (Ed.), Butterworth: London, 1968; 341–380.

