Random trials used in determining global minimum factors of safety of slopes

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The minimum factor of safety of a slope can be found by using various methods of optimization or by random search. This paper presents a combined approach that uses the random search to find an estimate of the global minimum which is employed by the methods of optimization as a starting point. Guidelines have been given to select an appropriate number of random trials. Two simplified methods for calculating the factors of safety are suggested to minimize the computing time of the random searches. The approach has proved to be more efficient than either a purely stochastic or a deterministic one.

Key words: slope stability, landslide, factor of safety, critical slip surface, method of optimization, random search, Monte Carlo method.

Le coefficient de sécurité minimum d'une pente peut être trouvé par différentes méthodes d'optimisation ou par recherche aléatoire. Cet article présente une approche combinée qui utilise la recherche aléatoire pour trouver une estimation du minimum global qui est utilisé par les méthodes d'optimisation comme point de départ. Des directives ont été données pour choisir un nombre approprié d'essais aléatoires. Deux méthodes simplifiées pour calculer le coefficient de sécurité sont suggérées pour minimiser le temps des recherches aléatoires. L'approche s'est révélée être plus efficace que celle purement stochastique ou déterministique.

Mots clés : stabilité des talus, glissement, coefficient de sécurité, surface de glissement critique, méthode d'optimisation, recherche aléatoire, méthode Monte Carlo.

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Introduction

Searching for minimum factor of safety in slope stability analysis has been made possible by the use of the methods of optimization (Baker 1980; Celestino and Duncan 1981; Nguyen 1985; Li and White 1987; Sun 1984; Chen and Shao 1988). Although the author's experience with this approach has been fairly successful, it also indicated that the algorithm occasionally suffered from premature termination at which the solution was actually not a global minimum of the factor of safety. Experience shows that the task of determining minimum factor of safety becomes harsh when (*i*) the problem involves many degrees of freedom, say seven or more; and (*ii*) the slope surface contains many zones of different soils. Figure 1*a* shows an example in which slip surface 1 was used as an initial estimate. The solution obtained by the method of optimization is represented by slip surface 3, and the critical slip surface is slip surface 2. Figure 1*b* shows a uniform slope with the same geometry and starting point as those of the case shown in Fig. 1*a*. (The geotechnical properties for the example of Fig. 1 are indicated in Table 1.) The calculation terminated successfully at the global minimum.

One of the most common philosophies dealing with the difficulties in finding the global minimum is that the nearer the initial estimate is to the final solution, the more promising it would be to have the algorithm succeed.

A problem that consequently arises is how to find an initial estimate that is close enough to the global minimum. An attractive approach is the technique of random search, whose main principle includes generating a set of variable vectors and examining their objective functions. The variable vector associated with the smallest objective function is retained as the initial estimate for various methods of optimization to minimize the factor of safety. The variable vectors are generated with the aid of random numbers. This technique allows a uniform, high-density scanning of the space containing the variable vectors and consequently a rough determination of the location of the critical slip surface.

Based on the principle of the Monte Carlo method (Hammersley and Handscomb 1964), it can be postulated that as the number of random searches approaches infinity, the smallest factor of safety will approach the global minimum. Therefore, the random search itself can be used to find the minimum factor of safety, as has been done by other researchers (Boutrup and Lovell 1980; Siegel *et al.* 1981). However, a very large number of searches will be needed to obtain sufficiently accurate solution, as will be discussed in this paper. It is for this reason that many authors (Fox 1971; Wolfe 1978; Shoup and Mistree 1987) suggested the use of random search only for the purpose of locating the initial estimate to be employed in other methods of optimization. The combined approach will prove to be superior to either a purely stochastic or a purely deterministic one.

The implementation of the random search

The conventional procedure of stability analysis

A typical procedure for determining the minimum factor of safety for a slope, such as that suggested by Chen and Morgenstern (1983) and Chen and Shao (1988), contains the following steps.

(1) Divide a slip surface by a number of nodal points $A_1, A_2, \ldots, A_m$ (Fig. 2) whose coordinates are

$$[1] \quad Z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

TABLE 1. Geotechnical properties of the example shown in Fig. 1

	Soil layer	Density ρ (g/cm^3)	Shear strength parameter ϕ' (°)	C' (kPa)
Fig. 1*a*	I	2.0	38	0
	II	2.0	23	5.3
	III	2.0	20	7.2
Fig. 1*b*	III	2.0	19.6	3.0

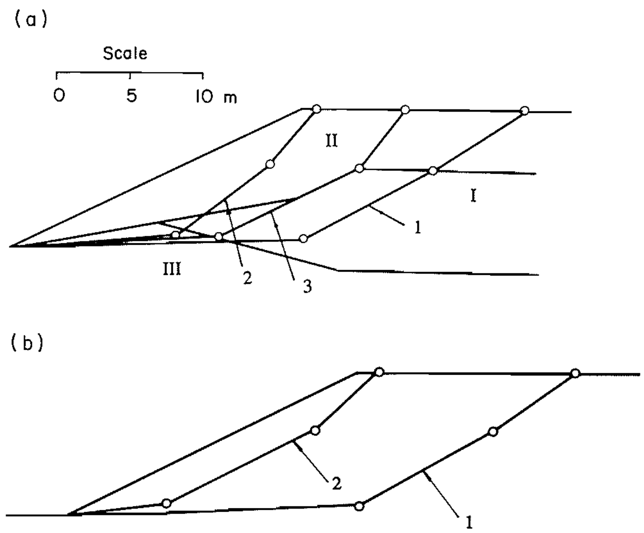

FIG. 1. An example displaying the success and failure of the searches for the minimum factor of safety. (*a*) A layered slope. Slip surfaces: 1, the initial estimate, $F_0 = 1.951$; 2, the known critical slip surface, $F_m = 1.484$; 3, the slip surface at which the calculation terminates, $F = 1.511$. (*b*) A uniform slope. Slip surfaces: 1, the initial estimate, $F_0 = 1.576$; 2, the critical slip surface at which the calculation terminates, $F_m = 1.006$.

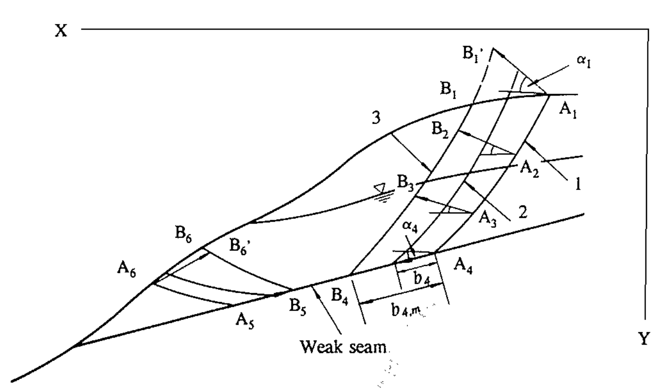

FIG. 2. Slip surfaces of generalized shape. 1, Z^0, the initial estimate; 2, Z, the slip surface during the process of optimization; 3, Z^m, the critical slip surface.

where $i = 1, 2, ..., m$. The slip surface is approximated by connecting each pair of contiguous nodal points with a straight line or a smooth curve. A smooth curve is generally preferred unless a weak band, such as A_4–A_5 in Fig. 2, is simulated.

The value of factor of safety F can be determined by the conventional method of slices (see Appendix). F is then expressed as a function with respect to $x_1, y_1, x_2, y_2, ..., x_m, y_m$.

[2] $F = F(x_1, y_1, x_2, y_2, ..., x_m, y_m)$

(2) For each nodal point specify a direction of movement α_i in which the point moves from A_i towards B_i, the corresponding nodal point that defines the critical slip surface Z^m (Fig. 2). The coordinates of the ith nodal points of any slip surface, Z_i, can be defined in terms of the relative increment b_i with respect to Z_i^0, a reference variable vector that is generally taken as the initial estimate defined during the optimization process:

[3] $$Z_i^0 = \begin{pmatrix} x_i^0 \\ y_i^0 \end{pmatrix}$$

[4] $$Z_i = Z_i^0 + \Delta Z_i$$

$$= Z_i^0 + b_i \begin{pmatrix} \cos \alpha_i \\ \sin \alpha_i \end{pmatrix}$$

where b_i is the distance along α_i, the specified direction of movement. The factor of safety can also be expressed as a function with respect to b_i ($i = 1, 2, ..., n$).

[5] $F = F(b) = F(b_1, b_2, ..., b_n)$

Some of the nodal points may be specified to be fixed, and n is the total number of the nodal points that are specified to move during the process of optimization; n is referred to as the degree of freedom.

The author's early work (Chen and Shao 1988) provides an option that allows a nodal point to move in both x and y directions rather than in a specified direction. This alternative has the disadvantage of having two degrees of freedom for each nodal point, which increases the numerical efforts in finding a converged solution. Since these efforts are seldom rewarded by more accurate solutions, the alternative was not used later. It is suggested that (*i*) for a nodal point located in a weak band, such as a discontinuity of a rock mass, α_i is taken to be the dip angle of that band (refer to points A_4 and A_5 in Fig. 2); and (*ii*) if the nodal points are located in an area in which the failure mode is not controlled by geological structures, specify α_i based on experience. Different specified values usually do not affect the final solution appreciably if these points are connected by smooth curves.

(3) Find a variable vector b_m associated with F_m, the minimum of F, by various methods of optimization, such as the Simplex method, Deviation–Fletcher–Powell (DFP) method, etc.

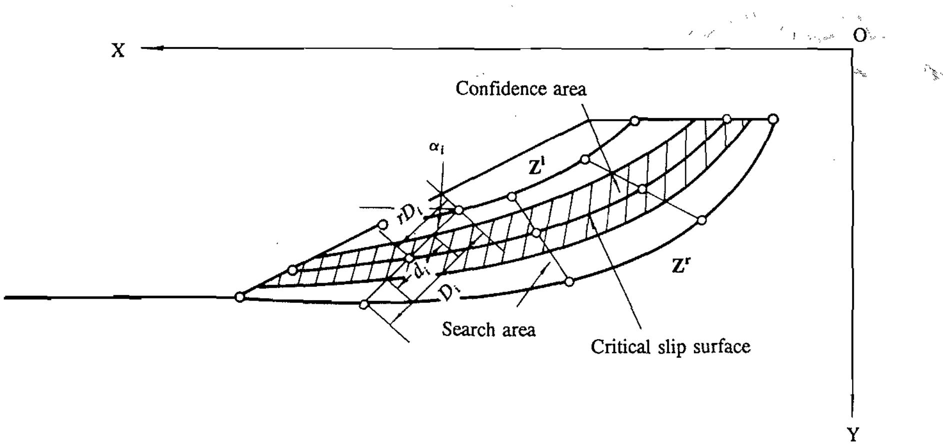

FIG. 3. The search area and confidence area.

The procedure of the random search

The purpose of random search is to find an initial estimate Z^0 used as the starting point in search of b_m, or Z^m, by conventional methods of optimization. The procedure is proposed to consist of the following steps.

(1) Define an area, called search area, that is expected to cover the critical slip surface being sought. This area, shown in Fig. 3, has its left and right borders Z^l and Z^r, respectively, each represented by a slip surface. The coordinates of the nodal points of Z^l and Z^r are

$$[6] \quad Z_i^l = \begin{pmatrix} x_i^l \\ y_i^l \end{pmatrix}$$

$$[7] \quad Z_i^r = Z_i^l + D_i \begin{pmatrix} \cos \alpha_i \\ \sin \alpha_i \end{pmatrix}$$

where $i = 1, 2, ..., m$ and D_i is called the band width of the search area.

$$[8] \quad D = (D_1, D_2, ..., D_n)^T$$

(2) Generate a slip surface Z^0 that falls into the search area and calculate its factor of safety F_0. The coordinates of Z^0 is determined by

$$[9] \quad Z_i^0 = Z_i^l + r_iD_i \begin{pmatrix} \cos \alpha_i \\ \sin \alpha_i \end{pmatrix}$$

where r_i is a random number ranging between 0 and 1. The generation of random numbers is a special topic covered by statistics textbooks. Most computer libraries provide pseudorandom number generators that offer reproducible random number sequences. These random numbers have been successfully subjected to various statistical tests. As a consequence, any part of the search area hold equal likelihood of accommodating Z^0, so determined by [9].

(3) Repeat step 2 and determine another slip surface Z' and its associated factor of safety F'. Compare F_0 with F'. Z^0 and F_0 are replaced by Z' and F', respectively, if F' is smaller than F_0.

(4) Step 3 is repeated and Z^0 and F_0 are renewed until the number of trials reached a specified value N and Z^0 is believed to be close enough to Z^m.

Determination of the number of random trials

It is clear that the greater the number of random trials, the more certain one may feel with the Z^0 so obtained. However, the computing time consumed by too many random trials can hardly be afforded by practitioners. The problem that consequently arises is, what is the general guideline to select an appropriate number of random trials? Based on the postulations originally proposed by Brooks (1958), and in the light of the special features a slope stability problem may possess, the following formulations are presented (refer to Fig. 3).

Suppose that in the neighbourhood of the critical slip surface there exists a band, called confidence area, in which any kinematically acceptable slip surface can be used as an initial estimate to conduct a successful search of the critical slip surface by the conventional methods of optimization. Its left and right borders are also represented by slip surfaces, as is done for the search area. It is further assumed that the band widths d_i of the confidence area are proportional to those of the search area (D_i) with a constant ratio m:

$$[10] \quad m = \frac{d_1}{D_1} = \frac{d_2}{D_2} = ... = \frac{d_n}{D_n}$$

By the theory of probability, the chance of successfully having all the n nodal points of a randomly generated slip surface fall into the confidence area is m^n. According to the binomial distribution theory (Lindley 1965) the probability of having r successes in N random surface is

$$[11] \quad P(^r_N) = C^r_N(m^n)^r(1 - m^n)^{N-r}$$

The probability that in N trials there is at least one success is

$$[12] \quad P = 1 - C^0_N(m^n)^0(1 - m^n)^N$$
$$= 1 - (1 - m^n)^N$$

P is called confidence level. It indicates the certainty with which one can expect to have at least one randomly generated slip surface fall into the confidence area if N random trials are performed.

Take that shown in Fig. 1*a* as an example. The slip surface of generalized shape is divided into three movable nodal points A, B, and C and a fixed one D as shown in Fig. 4. Based on experience, a search area is defined as the one shaded. It is presumed that the confidence ratio m is 0.5.

TABLE 2. Geotechnical properties of the example shown in Fig. 7

	Density ρ (g/cm^3)		Shear strength parameters	
Soil layer	Saturated	Wet	ϕ' (°)	C' (kPa)
Clay core	2.07	2.01	25	20
Rock fill	2.3	2.11	40	0
Weak seam	2.07	2.01	14	5

TABLE 3. Factor of safety obtained by various simplified methods and the rigorous method for the slip surfaces shown in Fig. 7

Slip surfaces	Simplified method 1	Lowe-Karafiath method	Simplified method 2	Rigorous method
$A_2B_2C_2D_2E_2$	1.501	1.500	1.726	1.769
$A_1B_1C_1D_2E_2$	1.276	1.256	1.417	1.446

NOTE: The Lowe-Karafiath method is found in Lowe and Karafiath (1959).

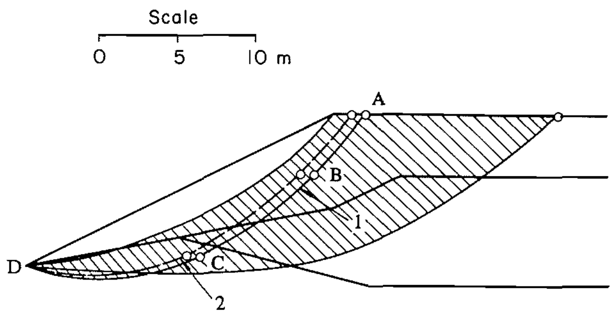

FIG. 4. Search for the minimum factor of safety for the problem shown in Fig. 1*a*. Slip surfaces: 1, the initial estimate, determined by the random search, F_0 = 1.384; 2, the critical, obtained by the Simplex method, F_m = 1.373.

From [12] it can be found that 25 trials are required to gain 96% confidence of success.

$$P = 1 - (1 - 0.5^3)^{25} = 96\%$$

On the other hand, from the viewpoint at conservativeness one may define a search area larger than the previous one. A smaller confidence ratio, say m = 0.3, is adopted then. To approximately reach the same confidence level 122 random trials will be needed.

$$P = 1 - (1 - 0.3^3)^{122} = 96\%$$

The search area and confidence area are determined based on experience and adjustment. It is usually not difficult to make the judgement. One or two reevaluations might be helpful to gain an insight into the approximate location of the critical slip surface and hence make an appropriate setting of the search area and the evaluation of m.

Restrictions to the shape of the random slip surfaces

The slip surfaces generated by [9] include some curves whose shapes are kinematically unacceptable as a failure surface, as discussed by Siegel *et al.* (1981). Some restrictions may be imposed based on experience. For example, we impose the following restrictions to Z calculated by [9] for soil slopes.

For any two contiguous nodal points i and i + 1, it is required that

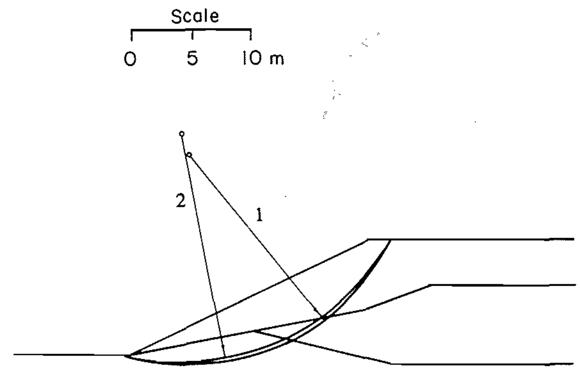

FIG. 5. Comparisons between the results obtained by the Bishop's simplified method and the simplified method 1 for the example shown in Fig. 1*a* (horizontal seismic force coefficient = 0.15*g*). The critical slip surfaces: 1, obtained by the Bishop's simplified method, F_m = 1.006; 2, obtained by the simplified method 1, F_m = 0.967.

[13] $x_{i+1} \geq x_i$

[14] $\psi_{i+1} \leq \psi_i$

where ψ_i is the inclination of the line connecting the points i and i + 1 to the x axis.

Any slip surface that violates either of the above restrictions is rejected immediately. The efficiency of the random search can be greatly enhanced with these restrictions.

Simplified methods used for random search

From [12] it can be found that the number of random trials N required for a successful search at a certain confidence level increases by the exponent of the degrees of freedom n. Several hundred or thousand trials may be needed if the problem involves 5–7 degrees of freedom. This demanding computing effort can be alleviated by using simplified methods to calculate factor of safety during random search. Since the purpose of the random search is only to locate an initial estimate, a high level of accuracy for factor of safety is unnecessary. In the Appendix, two simplified methods are proposed. They are straightforward, without the need of iteration; applicable to slip surfaces of generalized shape; and reasonably accurate. The simplified method 1

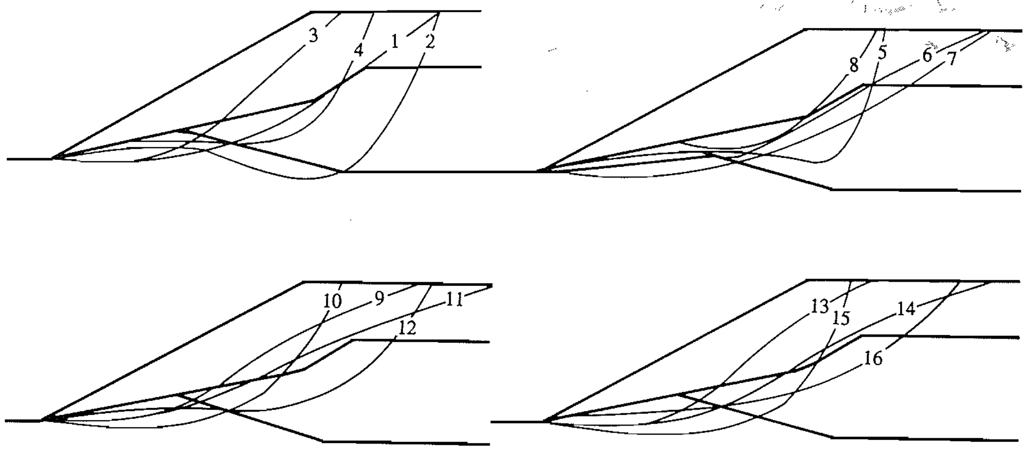

FIG. 6. The random slip surfaces generated in the search area shown in Fig. 5. Refer to Table 4 for factors of safety.

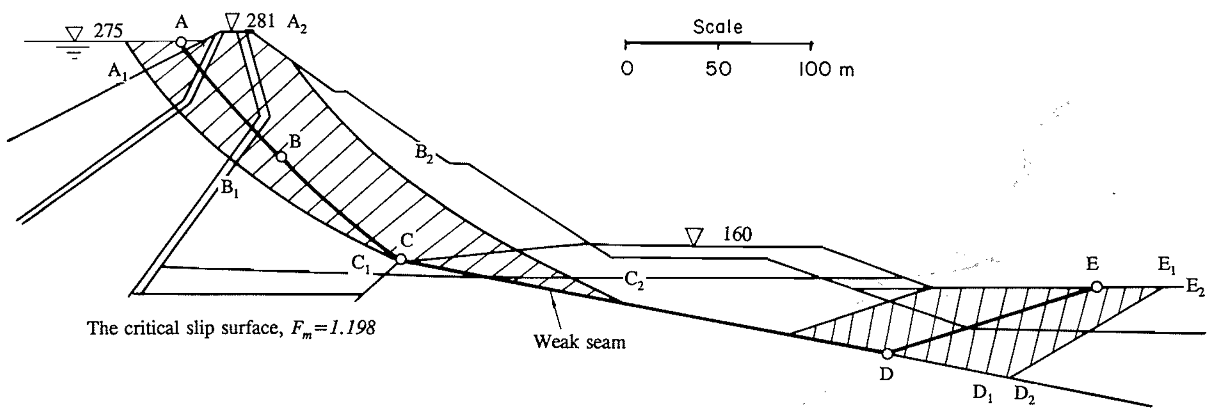

FIG. 7. Stability analysis of the downstream slope for the Xiaolangdi rock-fill dam.

was actually the suggested method to find the initial estimate F_1 in Chen and Morgenstern's (1983) approach but was not formally published then. It has been subjected to numerous tests and has proved to be a good method to calculate the approximate values of F provided that the slip surface is fairly smooth (Chen 1986). Figure 5 compares the critical slip surface obtained by the Bishop's simplified method with that obtained by the simplified method 1 for the case with horizontal seismic forces. Good agreement between the results of the two methods has been obtained.

It has been noted that for a slip surface containing some abrupt bendings, such as the points C and D in Fig. 7 (geotechnical properties of this example given in Table 2), the difference between the values of F of the accurate method and the various simplified methods can be significant. The simplified method 2, shown in the Appendix, is consequently suggested. This method actually performs the rigorous procedure but stops right after the first iteration ends. Since Chen and Morgenstern's (1983) approach rigorously follows the Newton–Raphson method, the first iteration usually does the most to improve the factor of safety when it is transferred from its initial estimate to the final solution. Table 3 presents the values of F obtained by various simplified methods and the rigorous method for the example shown in Fig. 7. It can be found that only the results given by the simplified method 2 were close enough to the accurate value of F.

TABLE 4. Factors of safety of the random surfaces shown in Fig. 6, obtained by the simplified method 1

Surface	F
1	1.509
2	3.375
3	1.373
4	1.861
5	3.459
6	1.686
7	1.580
8	1.535
9	1.657
10	1.412
11	1.931
12	1.987
13	1.449
14	1.696
15	1.529
16	1.774

Illustrative examples

Let us refer back to the failure to find the global minimum as discussed in the Introduction and shown in Fig. 1a, but this time points A, B, C, and D were connected by smooth curves. The search area was defined as the cross-

TABLE 5. The minimum factor of safety calculated by various methods for the example shown in Fig. 4

Method	Random search not involved		Random search involved		
	CPU time	F_m	CPU for random search	CPU, total	F_m
Simplex		(Fail)	39	421	1.368
Deviation–Fletcher–Powell		(Fail)	39	327	1.373
Powell	522	1.388	39	413	1.378

NOTE: Central processing unit (CPU) time is measured in seconds on an IBM-PC-XT.

TABLE 6. Search for the minimum factor of safety by various approaches (Fig. 7)

	N	CPU (time)	F_m
Purely random search	223	699	1.293
	605	3 431	1.270
	1336	6 437	1.261
	2356	11 170	1.253
Simplex method	0	2 940	1.234
Combined approach	223	2 661	1.199

NOTE: N, number of random searches involved. Central processing unit (CPU) time is measured in seconds on IBM-PC-XT. The combined approach performs 223 random searches, after which the Simplex method was used.

hatched area in Fig. 4, and m was taken to be 0.4. Using the simplified method 1, 45 random searches were performed. A confidence level of 95% can be obtained according to [12]. Figure 6 shows 16 of the 45 randomly generated slip surfaces and Table 4 shows the factors of safety of these surfaces, obtained by the simplified method 1. The one that is associated with the smallest factor of safety is shown by the solid line in Fig. 4. Starting from this slip surface, any optimization method can approach the critical slip surface without difficulty. Table 5 shows the calculated results and the computing time associated with the Simplex method, DFP method, and Powell's method. Without random search, both the Simplex and DFP methods failed to converge. The Powell's method was adopted lately by the author and is also aimed at overcoming the convergence difficulties. It involves some treatments particularly concerned with the slope stability problem. It did succeed in finding the global minimum without the random search but was more time consuming.

Figure 7 presents the practical applications of this approach. The 167 m high Xiaolangdi rock-fill dam in China is constructed on the Yellow River alluvium that overlies the limestone bedrock containing a number of clay intercalations. Stability of the downstream slope of the dam under the applications of seismic forces equivalent to a magnitude of 8 on Richter scale was investigated. The nodal points A, B, and C were connected by a smooth curve, whereas points C and D were connected by a straight line that simulates the weak intercalation. D and E were connected by a straight line. The search area is the cross-hatched area in Fig. 7. To obtain a confidence level of 90% (m = 0.4), 223 random trials were performed. The calculated critical slip surface was shown as $ABCDE$. Table 6 shows the calculated results for F_m and the computing time associated with various approaches. It can be found that in the case of purely random search, F_0 approaches F_m very slowly as the number of random searches N increases. It got F_m = 1.253 (CPU time 11 170 s) compared with F_m = 1.199 (CPU time 2661 s) obtained by the combined approach. On the other hand, the search by the Simplex method without the aid of random search gave a converged value of F_m but was less accurate and more time consuming than the approach involving the random search.

Summary and concluding remarks

(1) The critical slip surface of a slope can be determined by either the random search method or the methods of optimization. The computing effort of the former increased by the exponent of the degree of freedom, whereas the latter has the limitation of occasionally missing the global minimum factor of safety. The approach described in this paper combines the two methods and successfully circumvents their respective drawbacks.

(2) The method uses random search to find the rough location of the critical slip surface which is employed as the initial estimate of the method of optimization. The estimate so obtained will allow a quick approach to determination of the actual critical slip surface.

(3) The use of the simplified methods has enabled a very effective random search and hence minimized the computing time incurred by the random search.

(4) Admittedly, the approach described herein does not exclude the alternative of using the optimization method solely if the problem investigated is simple and numerical convergence for global minimum is not of concern.

Acknowledgement

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Appendix: Simplified methods of calculating factor of safety

The rigorous method

Allowing the presence of horizontal seismic forces, the force and moment equilibrium equations of the method of slices given by Chen and Morgenstern (1983) are updated here (refer to Fig. A1).

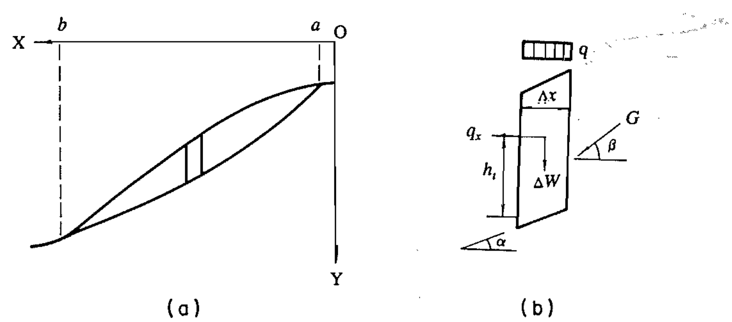

FIG. A1. The limiting equilibrium of a slope analyzed by the method of slices. (*a*) Stability analysis along a potential slip surface. (*b*) Forces applying on a slice.

[A1] $$G_n(F,\lambda) = \int_a^b p(x)s(x)\,\mathrm{d}x = 0$$

[A2] $$M_n(F,\lambda) = \int_a^b p(x)s(x)t(x)\,\mathrm{d}x - M_\mathrm{e} = 0$$

in which

[A3] $$p(x) = \left(\frac{\mathrm{d}W}{\mathrm{d}x} + q\right)\sin(\phi_\mathrm{e}' - \alpha) - r_\mathrm{u}\frac{\mathrm{d}W}{\mathrm{d}x}\sec\alpha\sin\phi_\mathrm{e}' + C_\mathrm{e}'\sec a\cos\phi_\mathrm{e}' - q_x\cos(\phi_\mathrm{e}' - \alpha)$$

[A4] $$s(x) = \sec(\phi_\mathrm{e}' - \alpha + \beta)\exp\left[-\int_a^x \tan(\phi_\mathrm{e}' - \alpha + \beta)\frac{\mathrm{d}\beta}{\mathrm{d}\zeta}\,\mathrm{d}\zeta\right]$$

[A5] $$t(x) = \int_a^x (\sin\beta - \cos\beta\tan\alpha)\exp\left[\int_a^\xi \tan(\phi_\mathrm{e}' - \alpha + \beta)\frac{\mathrm{d}\beta}{\mathrm{d}\zeta}\,\mathrm{d}\zeta\right]\mathrm{d}\xi$$

[A6] $$M_\mathrm{e} = \int_a^b q_x h_t\,\mathrm{d}x$$

[A7] $$\tan\beta = f_0(x) + \lambda f(x)$$

in which F is factor of safety; $\tan\phi'$ and C' are effective coefficients of friction and cohesion, respectively; $\tan\phi_\mathrm{e}'$ and C_e' are $(\tan\phi')/F$ and C'/F, respectively; $\mathrm{d}W/\mathrm{d}x$ is the weight of the slice per unit width;

r_u is pore-pressure ratio; q and q_x are vertical surface load and horizontal seismic force per unit width, respectively; h_t is distance between the point of action of q_x and the base of the slice; α and β are the inclinations of the base of the slice and the interslice force G to the horizontal, respectively; $f_0(x)$ and $f(x)$ are assumed distributions of $\tan\beta$; and λ is a coefficient to be determined.

[A1] and [A2] involve two unknowns, F and λ, which are solved using Newton–Raphson method by the following iterative procedure.

$$\text{[A8]}\quad \Delta F_i = F_{i+1} - F_i$$

$$\text{[A9]}\quad \Delta\lambda_i = \lambda_{i+1} - \lambda_i$$

The formulas calculating [A8] and [A9] and $\partial G_n/\partial F$, $\partial M_n/\partial F$, $\partial G_n/\partial\lambda$, and $\partial M_n/\partial\lambda$, which are involved in [A8] and [A9], are [42]–[51] found in Chen and Morgenstern (1983). Among them only [48] needs updating due to the presence of the horizontal seismic forces, which are given as [A10] here.

$$\text{[A10]}\quad k(x) = -\left[\left(\frac{dW}{dx} + q\right)\sin\phi_e'\cos\beta - r_u\frac{dW}{dx}\sec\alpha\sin\phi_e'\cos(\beta-\alpha) + C_e'\sec\alpha\cos\phi_e'\cos(\beta-\alpha) - q_x\sin\phi_e'\sin\beta\right]\cos\phi_e'/[F\cdot p(x)\cos(\phi_e'-\alpha+\beta)]$$

Assuming an initial estimate F_1 and λ_1, and substituting them into [8] and [9], F_2 and λ_2 can be obtained. The procedure is repeated until the specified convergence criterions are met. The initial estimate F_1 can be determined by the simplified method 1 given subsequently, and λ_1 is obtained by substituting [A7] into

$$\text{[A11]}\quad \int_a^b \tan\beta\,dx = \int_a^b \tan\alpha\,dx$$

Simplified method 1

Assuming that $\beta = \alpha$ and that the horizontal seismic force applies at the base of the slice, i.e.,

$$\text{[A12]}\quad h_t = 0$$

[A2] is then automatically satisfied, and [A1] results in the following equation to calculate factor of safety:

$$\text{[A13]}\quad F = \frac{\int_a^b A\cdot\exp\left[-\left(\frac{\tan\phi'}{F}\alpha + \frac{K_i}{F}\right)\right]dx}{\int_a^b B\cdot\exp\left[-\left(\frac{\tan\phi'}{F}\alpha + \frac{K_i}{F}\right)\right]dx}$$

where

$$\text{[A14]}\quad A = \left[\frac{dW}{dx}(\cos\alpha - r_u\sec\alpha)\tan\phi' + C'\sec\alpha - q_x\sin\alpha\tan\phi' + q\cos\alpha\tan\phi'\right]$$

$$\text{[A15]}\quad B = \left[\left(\frac{dW}{dx} + q\right)\sin\alpha + q_x\cos\alpha\right]$$

$$\text{[A16]}\quad K_i = \sum_{j=1}^{i}[\tan\phi_j'\,\alpha_j]_l^r$$

K_i is a coefficient accounting for possible abrupt change in α or ϕ' between two contiguous slices, and α_i is in radians. K_i is a constant in an interval of slip surface that is smooth and homogeneous and will be increased by $(\tan\phi_i'\cdot\alpha_i)_l^r$ after passing through the point i at which α_i or ϕ_i changes abruptly. $(\tan\phi_i'\cdot\alpha_i)_l^r$ is defined as $[(\tan\phi_i'\cdot\alpha_i)_r - (\tan\phi_i'\cdot\alpha_i)_l]$ where r and l refer to the right and left values of the variables in the parentheses at the point of discontinuity.

Since both sides of [A13] involve F, iteration is necessary for solving F. For further simplification, we introduce the following formulations.

First, add a constant K_c/F to the variables of the exponent functions in both the numerator and denominator of [A13]:

$$\text{[A17]}\quad F = \frac{\int_a^b A\cdot\exp\left[-\left(\frac{\tan\phi'}{F}\alpha + \frac{K_i}{F}\right) + \frac{K_c}{F}\right]dx}{\int_a^b B\cdot\exp\left[-\left(\frac{\tan\phi'}{F}\alpha + \frac{K_i}{F}\right) + \frac{K_c}{F}\right]dx}$$

It is not difficult to demonstrate that the right side of [A17] is identical to that of [A13]. K_c/F is selected that

$$- \left(\frac{\tan \phi'}{F}\alpha + \frac{K_i}{F}\right) + \frac{K_c}{F}$$

is small in absolute value compared with unity, so the following approximation stands:

$$[A18] \quad \exp\left[-\left(\frac{\tan \phi'}{F}\alpha + \frac{K_i}{F}\right) + \frac{K_e}{F}\right] = 1 - \left(\frac{\tan \phi'}{F}\alpha + \frac{K_i}{F}\right) + \frac{K_c}{F}$$

It is found that this requirement can be met basically for all the slices if we take

$$[A19] \quad K_c = \frac{\tan \phi'_{av} \cdot \alpha_{av}}{F}$$

where ϕ'_{av} and α_{av} are the average value of ϕ' and α between the inverval (a, b), respectively.

Second, substituting [A18] into [A17], we have

$$[A20] \quad F = \frac{B_k + (B_k^2 - 4A_kC_k)^{1/2}}{2A_k}$$

where

$$[A21] \quad A_k = \int_a^b B \, dx$$

$$[A22] \quad B_k = \int_a^b A \, dx + \int_a^b B\zeta \, dx$$

$$[A23] \quad C_k = \int_a^b A\zeta \, dx$$

$$[A24] \quad \zeta = \tan \phi' \cdot \alpha + K_i - \tan \phi'_{av} \cdot \alpha_{av}$$

Since C_k is generally small compared with B_k, [A18] can be further simplified as

$$[A25] \quad F = \frac{B_k}{A_k} - \frac{C_k}{B_k}$$

Equation [A25] is the formula calculating factor of safety by the simplified method 1.

Simplified method 2

This method performs the rigorous method but stops right after the first iteration finishes. Assumptions made for $f_0(x)$ and $f(x)$ in [A7] are

$$[A26] \quad f_0(x) = 0$$

and

$$[A27] \quad f(x) = 1$$